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LARGE SAMPLE PROPERTIES OF THE BAYES' SEQUENTIAL PROCEDURE FOR --ETC(U)

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LARGE SAMPLE PROPERTIES OF THE BAYES' SEQUENTIAL PROCEDURE FOR ESTIMATING THE ARRIVAL RATE OF A POISSON PROCESS WITH INVARIANT LOSS

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LARGE SAMPLE PROPERTIES OF THE BAYES' SEQUENTIAL PROCEDURE FOR ESTIMATING THE ARRIVAL RATE OF A POISSON PROCESS WITH INVARIANT LOSS

C. P. Shapiro† and Robert Wardrop‡

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ABSTRACT

Let W_n, $n = 0,1,\ldots$ be the time until the nth arrival of a Poisson process with rate θ. Using invariant loss $L(\theta, \hat{\theta}) = \theta^{-2}(\theta-\hat{\theta})^2$ and sampling costs involving cost per arrival and cost per unit time, the Bayes' sequential procedure $(N^*, \hat{\theta}_{N^*})$ is derived. The large sample properties of the procedure are then studied in the classical framework, and N^*, the stopping time, is shown to be asymptotically equivalent to n^*, the best fixed sample size procedure when θ is known. Asymptotic normality of the sequential estimator $\hat{\theta}_{N^*}$ is also shown.

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†This work was done while at the University of Wisconsin, Madison and on leave from Michigan State University.

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LARGE SAMPLE PROPERTIES OF THE BAYES' SEQUENTIAL

PROCEDURE FOR ESTIMATING THE ARRIVAL RATE OF A POISSON PROCESS

WITH INVARIANT LOSS

C. P. Shapiro[†] and Robert Wardrop[‡]

1. Introduction. Let W_n, $n = 0,1,\ldots$, be the time until the nth arrival of a Poisson process with rate θ. Take $W_0 = 0$. Conditional on θ, W_n has a gamma distribution with shape parameter n and mean n/θ (gamma(n,θ)). Let $\mathcal{F}_n$, $n = 0,1,\ldots,$ denote the sigma algebra generated by W_i, $0 \leq i \leq n$. Sequential estimation procedures of the form $(N,\hat{\theta}_N)$ are considered, where N, the number of arrivals observed, is a stopping time with respect to $\mathcal{F}_n$, and $\hat{\theta}_N$ is an $\mathcal{F}_N$ measurable random variable, with $\mathcal{F}_N$ the sigma algebra of events prior to N.

The loss due to estimation is $L(\theta,\hat{\theta}) = \theta^{-2}(\theta-\hat{\theta})^2$. Using this loss, the decision problem of estimation of θ is invariant under the group of scale transformations (Ferguson, 1967). Such a loss function measures the estimation error in variance units, θ^{-2}, and forces more precision at small values of θ.

The cost of sampling involves two components: c_A = the cost of observing one arrival, and c_T = the cost of observing the process for one unit of time.

In Section 2, the Bayes' sequential procedure (denoted $(N^*,\hat{\theta}_{N^*})$ throughout the paper) is derived in Theorem 2.1 using a gamma prior on θ and the loss and cost structures described above. In Sections 3 and 4 the large sample properties of the procedure $(N^*,\hat{\theta}_{N^*})$ are examined without reference to the Bayesian origin of the procedure. The limiting form of N^* is given in Theorem 3.1 and the asymptotic normality of $\hat{\theta}_{N^*}$ is given in Theorem 4.1.

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2. The Bayes' Sequential Procedure. The problem of finding the Bayes' procedure $(N^*, \hat{\theta}_{N^*})$ is solved as follows. For a given stopping time N, $\hat{\theta}_N$ is the Bayes' estimator of θ given F_N. The optimal choice of N is then obtained by finding that stopping time which minimizes the expected total cost (Bayes' risk due to estimation plus the expected cost of sampling). See DeGroot (1970) or Chow, Robbins, and Siegmund (1971) for more details.

Suppose θ has prior distribution gamma (α_0, β_0) where $\alpha_0 \geq 2$ and $\beta_0 > 0$. Then the posterior distribution of θ given F_n is gamma (α_n, β_n) with $\alpha_n = \alpha_0 + n$ and $\beta_n = \beta_0 + W_n$. Using the loss function given in Section 1, the Bayes' estimator of θ given F_n is $\hat{\theta}_n = \beta_n^{-1}(\alpha_n - 2)$, and the expected posterior loss using $\hat{\theta}_n$ is $E[L(\theta, \hat{\theta}_n) | F_n] = (\alpha_n - 1)^{-1}$. Thus, by the strong Markov property, the total cost of the procedure $(N, \hat{\theta}_N)$ is

$$C_N = (\alpha_N - 1)^{-1} + c_A N + c_T W_N .$$

The Bayes' procedure minimizes $E(C_N)$.

Define stopping rule $N^* =$ first $n \geq 0$ such that $c_T \beta_n + c_A \alpha_n \geq \alpha_n^{-1} + c_A$. Note that $P(N^* < \infty) = 1$, and that the rule is easy to use since both α_n and β_n have a simple form. Also, note that if $c_T = 0$, then N^* is a fixed sample size stopping rule since only β_n is random in the defining expression for N^*. Due to this degeneracy, assume $c_T > 0$ henceforth.

Lemma 2.1.

i) If $c_A > 0$, then $N^* \leq c_A^{-1/2}$,

ii) if $c_T > 0$, then $N^* \leq (\beta_0 c_T)^{-1}$.

Proof. Define stopping rules N_A and N_T by

$$N_A = \text{first } n \geq 0 \text{ such that } c_A \alpha_n \geq \alpha_n^{-1} + c_A,$$

$$N_T = \text{first } n \geq 0 \text{ such that } c_T \beta_n \geq \alpha_n^{-1} .$$

Then $N^* \leq \min(N_A, N_T)$. From the definition of N_A, either $N_A = 0$ or $N_A - 1$ satisfies the reverse inequality: $c_A \alpha_{N_A - 1} < \alpha_{N_A - 1}^{-1} + c_A$. This last expression implies that $N_A \leq c_A^{-1/2}$. A similar argument applied to N_T gives $N_T < (\beta_0 c_T)^{-1}$.

The following lemma is a technical result needed in the proof of Theorem 2.1.

Lemma 2.2. If N' is a stopping rule such that $Ec_{N'} < \infty$ then

$$\lim_{n \to \infty} \int_{\{N' > n\}} c_n dP = 0 .$$

Proof:

$c_n = (\alpha_n - 1)^{-1} + c_A n + c_T W_n$. Thus, $E\, c_{N'} < \infty$ if and only if $EN' < \infty$ and $EW_{N'} < \infty$. Consider each of the three c_n terms separately.

(i) $\int_{\{N'>n\}} (\alpha_n - 1)^{-1} dP \le (\alpha_n - 1)^{-1} P[N' > n] \to 0$ as $n \to \infty$.

(ii) $\int_{\{N'>n\}} c_A n dP = c_A \sum_{k=n+1}^{\infty} \int_{\{N'=k\}} n dP \le c_A \sum_{k=n+1}^{\infty} kP(N'=k)$.

This last term tends to 0 as $n \to \infty$ since $EN' < \infty$.

(iii) $\int_{\{N'>n\}} c_T W_n dP = c_T \sum_{k=n+1}^{\infty} \int_{\{N'=k\}} W_n dP \le c_T \sum_{k=n+1}^{\infty} \int_{\{N'=k\}} W_k dP$

since $W_n \le W_k$ for $k \ge n$.

This last term tends to 0 as $n \to \infty$ since $EW_{N'} < \infty$.

The theorem below states that the decision procedure $(N^*, \hat{\theta}_{N^*})$ minimizes the expected total cost among all decision procedures $(N', \hat{\theta}_{N'})$ with $N' \ge 0$, and thus $(N^*, \hat{\theta}_{N^*})$ is the Bayes' procedure.

Theorem 2.1. If $(N', \hat{\theta}_{N'})$ is a sequential decision procedure then $E(c_{N^*}) \le E(c_{N'})$.

Proof. The cost sequence is first shown to be in the monotone case. Note that given θ, $W_{n+1} = W_n + X$, where X is exponential (θ) and independent of W_n. Thus,

$$E(c_{n+1} | F_n, \theta) = (\alpha_{n+1} - 1)^{-1} + c_A(n+1) + c_T(W_n + \theta^{-1}).$$

Taking $E[\cdot | F_n]$ and using $E(\theta^{-1} | F_n) = \beta_n(\alpha_n - 1)^{-1}$ yields

$$E(c_{n+1} | F_n) = c_n + \alpha_n^{-1} - (\alpha_n - 1)^{-1} + c_A + c_T(\alpha_n - 1)^{-1}\beta_n.$$

Thus, the cost sequence is in the monotone case and the rule N^* can be expressed as $N^* =$ first $n \ge 0$ such that $E(c_{n+1} | c_n) \ge c_n$. Now since $c_T > 0$, Lemma 2.1 implies $Ec_{N^*} < \infty$. Let N' be any stopping time such that $N' \ge 0$. If $Ec_{N'} = \infty$, then N^* is obviously better. If $Ec_{N'} < \infty$, then Lemma 2.2 allows application of the monotone case theorem (Chow, Robbins, Siegmund, 1971) to conclude $Ec_{N^*} \le Ec_{N'}$.

3. **Large sample properties of N^*.** In this section the stopping rule N^* is examined in the classical framework. The parameter θ is considered fixed but unknown and all probabilities and expectations are conditional on θ and denoted P_θ, E_θ, respectively. The procedure $(N^*, \hat{\theta}_{N^*})$ does not minimize $E_\theta C_N$ for all θ, but only the average of $E_\theta C_N$ over the prior distribution of Section 2.

The large sample properties of the procedure $(N^*, \hat{\theta}_{N^*})$ are studied by letting the sampling costs tend to zero. Note that the stopping rule N^* is a function of the sampling costs $\underline{c} = (c_A, c_T)$. Define $n^* = n^*(\theta) = (c_A + c_T\,\theta^{-1})^{-1/2}$. The main result in this section (Theorem 3.1) is that N^* is asymptotically equivalent to n^* as $\underline{c}$ tends to $\underline{0} = (0,0)$ with $c_A c_T^{-1}$ tending to $c_0 \leq \infty$. As motivation for this limiting form of N^*, compute $E_\theta C_n$ equal to $(\alpha_n - 1)^{-1} + c_A n + n\,c_T\,\theta^{-1}$, where the expectation is conditional on θ. Let $H(x) = (\alpha_x - 1)^{-1} + c_A x + c_T x\,\theta^{-1}$. Then $H(x)$ attains a unique minimum at $x = (c_A + c_T\theta^{-1})^{-1/2} + (\alpha_0 - 1)$. Ignoring the $(\alpha_0 - 1)$ term, this minimum is n^* defined above.

The following lemmas give rates and uniform integrability results needed in the proof of Theorem 3.1. Two cases are considered depending on the limit, c_0, of $c_A c_T^{-1}$.

Lemma 3.1. For each $\varepsilon > 0$,

$$P_\theta\left(\left|\frac{N^*}{n^*} - 1\right| > \varepsilon\right) \leq b(\underline{c})\exp[c_T^{-1/2}\,D(\underline{c},\varepsilon)],$$

where $b(\underline{c}) \to b_0 < \infty$, and $D(\underline{c},\varepsilon) \to D(\varepsilon) < 0$ and finite, as $c_A, c_T \to 0$ such that $c_A c_T^{-1} \to c_0 < \infty$.

Proof: $P_\theta\left(\frac{N^*}{n^*} - 1 > \varepsilon\right) = P_\theta(W_k < B_k)$, where

W_k = waiting time until the k^{th} arrival, $B_k = (\alpha_k c_T)^{-1} - c_A c_T^{-1}(\alpha_k - 1) - \beta_0$, and $k = [(1+\varepsilon)n^*]$ with $[\cdot]$ the greatest integer function.

For all $t > 0$, $P_\theta(W_k < B_k) = P_\theta(\exp(-tW_k) > \exp(-t\,B_k)) \leq \exp(t\,B_k)\,E_\theta \exp(-tW_k)$

$$= \exp(t\,B_k - k\,\ln(1 + t\,\theta^{-1}))$$

by Bernstein's Inequality. Since $[x] \geq x-1$, the exponent is $\leq b^+(\underline{c}) + c_T^{-1/2} D^+(t,\underline{c},\varepsilon)$, where

$$b^+(\underline{c}) = -c_A c_T^{-1}\, t(\alpha_0 - 2) + \ln(1+t\theta^{-1}) - \beta_0 t \quad \text{and}$$

$$D^+(t,\underline{c},\varepsilon) = \frac{t(c_A c_T^{-1}\,\theta + 1)^{1/2}}{\theta^{1/2}(1+\varepsilon)} - \frac{c_A c_T^{-1} t\theta^{1/2}(1+\varepsilon)}{(c_A c_T^{-1}\theta + 1)^{1/2}} - \frac{\theta^{1/2}(1+\varepsilon)\,\ln(1+t\theta^{-1})}{(c_A c_T^{-1}\,\theta + 1)^{1/2}} .$$

As $c_A c_T^{-1} \to c_0 < \infty$, $D^+(t,\underline{c},\varepsilon) \to D^+(t,\varepsilon) > -\infty$ for all t. But $D^+(t,\varepsilon) < 0$ if and only if $\theta(1+\varepsilon)^2 > \dfrac{\theta + c_0^{-1}}{1+c_0^{-1}t^{-1}\ln(1+t\theta^{-1})}$. The right hand side tends to θ as $t \to 0$. Thus there exists t^+ such that $D^+(t,\varepsilon) < 0$ for all $t \leq t^+$.

A similar argument yields $P_\theta(\frac{N^*}{n^*} - 1 < -\varepsilon) \leq \exp(b^-(\underline{c}) + c_T^{-1/2} D^-(t,\underline{c},\varepsilon))$, where $D^-(t,\underline{c},\varepsilon) \to D^-(t,\varepsilon) > -\infty$, and $D^-(t,\varepsilon) < 0$ for all $t \leq t^-$, for some t^-. The proof is completed by setting $t_0 = \min(t^+, t^-)$, $D(\underline{c},\varepsilon) = \max(D^+(t_0,\underline{c},\varepsilon), D^-(t_0,\underline{c},\varepsilon))$, and $b(\underline{c}) = 2\max(\exp(b^-(\underline{c})), \exp b^+(c))$.

Lemma 3.2. For each ε, $0 < \varepsilon < 1$

$$P_\theta(|\frac{N^*}{n^*} - 1| > \varepsilon) \leq b(\underline{c})\exp c_A^{1/2}\, c_T^{-1} D(\underline{c},\varepsilon),$$

where $b(\underline{c}) \to b_0 < \infty$, and $D(\underline{c},\varepsilon) \to D(\varepsilon) < 0$ and finite, as $c_T, c_A \to 0$ such that $c_A c_T^{-1} \to \infty$.

Proof: The techniques here are similar to those of Lemma 3.1. $P_\theta(\frac{N^*}{n^*} - 1 < -\varepsilon) = P_\theta(W_k > B_k)$, where B_k is defined in the proof of Lemma 3.1 and $k = [(1-\varepsilon)n^*]$. Following Lemma 3.1, for all $t < \theta$ $P_\theta(W_k > B_k) \leq \exp(-t\, B_k - k\,\ln(1 - t\theta^{-1}))$, $\leq \exp(b^-(\underline{c}) + c_A^{1/2}\, c_T^{-1}\, D^-(t,\underline{c},\varepsilon))$ where $b^-(\underline{c}) = -\ln(1 - t\theta^{-1}) + \beta_0 t$ and

$$D^-(t,\underline{c},\varepsilon) = \frac{-t(\theta + c_T c_A^{-1})^{1/2}}{\theta^{1/2}(1-\varepsilon) + (\alpha_0+1)(c_A\theta + c_T)^{1/2}} + t c_A^{1/2}\alpha_0 + \frac{\theta^{1/2}(1-\varepsilon)(t - c_T c_A^{-1}\, \ln(1 - t\theta^{-1})}{(\theta + c_A c_T^{-1})} .$$

As $c_A, c_T \to 0$ and $c_A c_T^{-1} \to \infty$, $D^-(t,\underline{c},\varepsilon) \to D^-(t,\varepsilon) < 0$ and finite for all $\varepsilon < 1$. Similar methods applied to $P_\theta(\frac{N^*}{n^*} - 1 > \varepsilon)$ complete the proof.

Lemma 3.3. If $c_A, c_T \to 0$ such that $c_A c_T^{-1} \to c_0 \leq \infty$, then

(i) N^*/n^* is uniformly integrable (P_θ), and

(ii) n^*/N^* is uniformly integrable (P_θ).

Proof:

(i) Take $a > 1 + \varepsilon$. Suppose $c_0 < \infty$. Then Lemma 3.1 implies

$$\int_{\{N^*/n^* > a\}} N^*/n^* dP_\theta \leq (\beta_0 c_T)^{-1} P_\theta(N^*/n^* > 1 + \varepsilon)$$

$$\leq (\beta_0 c_T)^{-1} b(\underline{c}) \exp(c_T^{-1/2} D(\underline{c}, \varepsilon)),$$

which tends to zero as $\underline{c} \to 0$. If $c_0 = \infty$, then $N^*/n^* \leq (1 + c_T c_A^{-1} \theta^{-1})^{1/2}$. Thus, since $c_T c_A^{-1} \to 0$, N^*/n^* is uniformly bounded in $\underline{c}$ and hence uniformly integrable.

(ii) $N^* \geq 1$ implies $n^*/N^* \leq n^*$. Thus,

$$\int_{\{n^*/N^* > a\}} n^*/N^* dP_\theta \leq n^* P(n^*/N^* > a) = n^* P_\theta(N^*/n^* < a^{-1}).$$

Take $a > (1-\varepsilon)^{-1}$. Then if $c_A c_T^{-1} \to c_0 < \infty$, Lemma 3.1 implies the last expression is

$$\leq (c_A c_T^{-1} + \theta^{-1})^{-1/2} c_T^{-1/2} b(\underline{c}) \exp(c_T^{-1/2} D(\underline{c}, \varepsilon)),$$

which tends to 0. If $c_A c_T^{-1} \to \infty$, then Lemma 3.2 implies the last expression is

$$\leq (c_A^{-1} c_T)(1 + c_A^{-1} c_T \theta^{-1})^{-1/2} (c_A^{1/2} c_T^{-1}) b(\underline{c}) \exp(c_A^{1/2} c_T^{-1} D(\underline{c}, \varepsilon)),$$

which tends to 0.

Theorem 3.1. If $c_A, c_T \to 0$ such that $c_A c_T^{-1} \to c_0 \leq \infty$, then

(i) $N^*/n^* \to 1$ a.s. (P_θ),

(ii) $E_\theta \frac{N^*}{n^*} \to 1$.

Proof: (ii) follows from (i) and the uniform integrability shown in Lemma 3.3.

To prove (i), a Borel Cantelli type argument is used along with monoticity properties of N^* and n^*. Let $\underline{c}(k)$ be a sequence of costs decreasing coordinatewise to $\underline{0}$ such that $c_A(k)\, c_T(k)^{-1}$ tends to $c_0 \leq \infty$ as $k \to \infty$. For simplicity, let N_k^* and n_k^* denote N^*, n^* respectively when cost $\underline{c}(k)$ is used. From the definitions of N^* and n^*,

$$N_k^* \leq N_c^* \leq N_{k+1}^* \quad \text{and} \quad n_k^* \leq n_c^* \leq n_{k+1}^*,$$

for all $\underline{c}$ in $[\underline{c}(k+1), \underline{c}(k)]$ where containment is coordinatewise. Fix $\varepsilon > 0$.

Then $P_\theta(N^*/n^* > 1 + \varepsilon$ for some $\underline{c} \leq \underline{c}(m))$ is

$$\leq \sum_{k=m}^{\infty} P_\theta(N^*/n^* > 1+\varepsilon \text{ for some } \underline{c} \text{ in } [\underline{c}(k+1), \underline{c}(k)])$$

$$\leq \sum_{k=m}^{\infty} P_\theta\left(\frac{N^*_{k+1}}{n^*_{k+1}} > (1+\varepsilon)(n^*_k/n^*_{k+1})\right).$$

Since n^*_k/n^*_{k+1} tends to 1, choose $\varepsilon' = \varepsilon/2(1+\varepsilon)$, and m such that $n^*_k/n^*_{k+1} > 1-\varepsilon'$ for all $k \geq m$. Then the probability above is

$$\leq \sum_{k=m}^{\infty} P_\theta\left(\frac{N^*_{k+1}}{n^*_{k+1}} > 1 + (\varepsilon/2)\right)$$

which tends to zero as $m \to \infty$ from the exponential rates derived in Lemmas 3.1 and 3.2.

The limiting form of $E_\theta C_{N*}$ can be derived as a corollary to Theorem 3.1.

Corollary 3.1. If $c_A, c_T \to 0$ such that $c_A c_T^{-1} \to c_0 \leq \infty$, then

$$n^* E_\theta C_{N*} \to 2.$$

Proof: $E_\theta C_{N*} = E_\theta(\alpha_{N*}-1)^{-1} + (c_A + c_T\theta^{-1})E_\theta N^*$, and

$$n^* E_\theta C_{N*} = E_\theta\, n^*(\alpha_{N*}-1)^{-1} + E_\theta N^*/n^* .$$

The last term tends to 1 by Theorem 3.1. Also, by Theorem 3.1 $n^*(\alpha_{N*}-1)^{-1} \to 1$ a.s. (P_θ). But $n^*(\alpha_{N*}-1)^{-1} \leq n^*/N^*$ which is uniformly integrable by Lemma 3.3. Thus, the first term tends to 1.

4. Asymptotic normality of $\hat{\theta}_{N^*}$ and concluding remarks. Once the limiting form of N^* is found, asymptotic properties of $\hat{\theta}_{N^*}$ can be obtained by standard methods.

Lemma 4.1. Suppose $X_1, X_2, \ldots$ are independent and identically distributed with mean 0 and variance 1, and that N' is a stopping time tending to ∞ as sampling costs tend to 0. If there exists n', nonrandom, such that $N'/n' \to 1$ (in probability) then

$$Y_{N'} = \frac{1}{(N')^{1/2}} \sum_{1}^{N'} X_i \to Z \text{ (in distribution)},$$

where Z is normal with mean 0 and variance 1.

Proof: The result is well known (Renyi, 1957).

Theorem 4.1. If $c_A, c_T \to 0$ such that $c_A c_T^{-1} \to c_0 \leq \infty$, then

$$(N^*)^{1/2} \frac{(\hat{\theta}_{N^*} - \theta)}{\theta} \to Z \text{ (in distribution)}$$

where Z is normal with mean 0 and variance 1.

Proof:

$$P_\theta((N^*)^{1/2} \frac{(\hat{\theta}_{N^*} - \theta)}{\theta} \leq x)$$

$$= P_\theta(W_{N^*} \geq (N^*)^{1/2}(\alpha_0 - 2 + N^*)\theta^{-1}(x + (N^*)^{1/2})^{-1} - \beta_0)$$

$$= P_\theta(Y_{N^*} \geq -x + R(N^*)),$$

where

$$Y_{N^*} = \frac{\theta}{(N^*)^{1/2}} (W_{N^*} - N^*\theta^{-1})$$

and

$$R(N^*) = (\alpha_0 - 2)(x + (N^*)^{1/2})^{-1} - (N^*)^{1/2} x (x + (N^*)^{1/2})^{-1} + x - \beta_0 \theta (N^*)^{-1/2}.$$

Fix $\epsilon > 0$. Then $P_\theta(Y_{N^*} \geq -x + R(N^*))$ is

$$\leq P_\theta(Y_{N^*} \geq -x - \epsilon) + P_\theta(R(N^*) < -\epsilon).$$

From Lemma 4.1, $Y_{N^*} \to Z$ (in distribution), and thus the first term above tends to $1 - \Phi(-x-\epsilon) = \Phi(x+\epsilon)$, where $\Phi(\cdot)$ is the distribution function of a standard normal random variable. But $N^*/n^* \to 1$ (a.s. P_θ) and $n^* \to \infty$, implies $R(N^*) \to 0$ (a.s.) and hence, $P_\theta(R(N^*) < -\epsilon) \to 0$. Noting that ϵ is arbitrary completes the proof.

Although Bayesian methods are used to derive the procedure $(N^*, \hat{\theta}_{N^*})$, the procedure has desirable properties in the classical framework, and these properties are independent of the prior distribution in Section 2. In particular, n^* is approximately the best fixed sample size procedure if θ is known. Thus, the asymptotic equivalence of N^* and n^* is a strong result. Once this equivalence is proven, usual properties of fixed sample size estimators will hold for the sequential estimator of θ (as shown in Theorem 4.1).

The inclusion of two type of costs in this problem is much more realistic than the simple cost per arrival. Also, when only cost per arrival is considered, the best sequential procedure is a fixed sample size procedure. However, with the inclusion of time cost, the best procedure is no longer a fixed sample size procedure, and Theorem 3.1 shows how these two costs are weighted asymptotically in determining the sample size.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let W_n, n=0,1,..., be the time until the nth arrival of a Poisson process with rate θ. Using invariant loss $L(\theta,\hat{\theta}) = \theta^{-2}(\theta-\hat{\theta})^2$ and sampling costs involving cost per arrival and cost per unit time, the Bayes' sequential procedure $(N^*, \hat{\theta}_{N^*})$ is derived. The large sample properties of the procedure are then studied in the classical framework, and N^*, the stopping time, is shown to be asymptotically equivalent to n^*, the best fixed sample size procedure when θ is known. Asymptotically normality of the sequential estimator $\hat{\theta}_{N^*}$ is also shown.

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